Math 429 - Exercise Sheet 11

1. Explicitly write down the Cartan matrices corresponding to $\mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ (based on the simple roots you worked out last time) and check that the corresponding Dynkin diagrams are indeed B_n, C_n, D_n.

2. For any root system $R \subset U$, show that

$$R^\vee = \left\{ \alpha^\vee(-) = \frac{2(-,\alpha)}{(\alpha,\alpha)} \Big| \alpha \in R \right\} \subset U^*$$

is also a root system (we use the fact that the inner product identifies $U \cong U^*$). Show that if $\alpha_1, \dots, \alpha_r$ is a set of simple roots of R, then $\alpha_1^\vee, \dots, \alpha_r^\vee$ is a set of simple roots of $R^\vee$.

3. Show that the Weyl group of an irreducible root system acts transitively on the set of roots of given length (i.e. for any pair of short/long roots, there is a Weyl group element sending one to the other). *Hint: use the fact that $W \curvearrowright E$ is an irreducible representation (and prove this fact).*

4. Fill in the following gap in the proof of Theorem 19: if a Dynkin diagram consists of a triple edge connected to a double or triple edge, then the corresponding 3×3 matrix S (see Definition 23) is not positive-definite.

(*) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let G be the corresponding simply connected complex Lie group. For any root α (corresponding to a henceforth fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$), construct an element $S_\alpha \in G$ such that

$$\mathrm{Ad}_{S_\alpha} : \mathfrak{g}^* \to \mathfrak{g}^*$$

sends $\mathfrak{h}^*$ to $\mathfrak{h}^*$ and coincides with the simple reflection s_α. *Hint: deal first with the case $\mathfrak{g} = \mathfrak{sl}_2$.*